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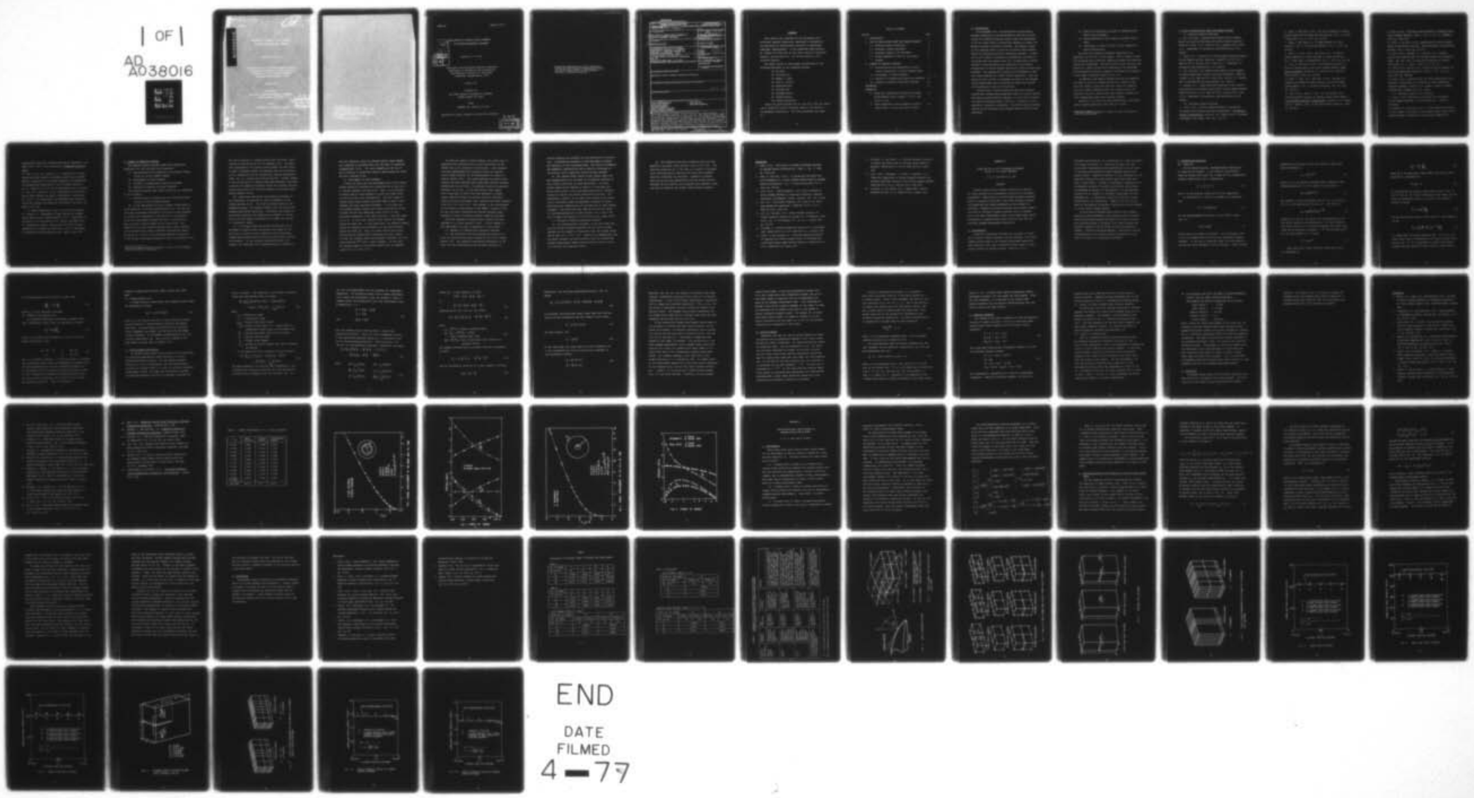

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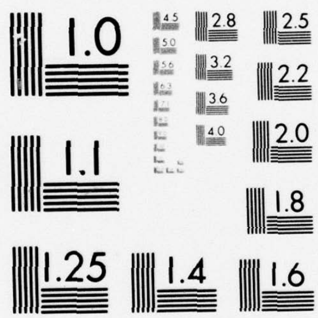

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APPLICATION OF HYBRID FINITE ELEMENTS TO STRUCTURE MECHANICS PROBLEMS

Theodore H. H. Pian

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Department of Aeronautics and Astronautics
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Cambridge, Massachusetts 02139

January 1977

Prepared for

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
UNITED STATES AIR FORCE

Under

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STRUCTURAL MECHANICS

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report presents a summary of the results of this research program. A list of all publications under this contract is given. The research findings summarized here are listed under four categories: (1) rationalization and extensions of variational formulations of finite element methods, (2) development of crack elements, (3) analysis of geometrically nonlinear problems, and (4) analysis of material nonlinear problems. Two appendices which contain more detailed descriptions are (A) Creep and Viscoplastic Analysis by Assumed Stress Hybrid Finite Elements and (B) Three-Dimensional Crack Elements by Hybrid Stress Model.

FOREWORD

This research was conducted by the Aeroelastic and Structures Research Laboratory, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts. It was supported under Contract No. F44620-72-C-0018 with the Air Force Office of Scientific Research, Bolling AFB, D.C. Mr. William Walker is the Technical Monitor.

The author wishes to acknowledge contributions of the following personnel to this research program:

Dr. Pin Tong

Dr. Chih-Hung Luk

Mr. Samuel J. Lasry

Dr. Peter L. Boland

Dr. Sheng-Taur Mau

Mr. Sung Won Lee

Mr. Kazumasa Moriya

Mr. Tasuku Shimizu

Mr. Thomas Scharnhorst

During the period August 1974 to June 1975, when the author was on sabbatical leave, Professor James W. Mar served as the principal investigator. The final manuscript was typed by

TABLE OF CONTENTS

I. Introduction

Since December 1971, the Aeroelastic and Structures Research Laboratory of the Massachusetts Institute of Technology has conducted research studies under the sponsorship of the Air Force Office of Scientific Research on static and dynamic nonlinear structural problems. The general thread of this research effort is the use of hybrid finite element models for analyzing structural mechanics problems. In addition to the study of problems with geometrical and material nonlinearities, the present task also includes basic investigations of rationalization and extensions of variational formulations of finite element methods for linear and nonlinear problems. The program also contains the development of special crack elements for linear fracture mechanics. Such elements for both 2-D and 3-D problems are based on the assumed stress hybrid model and are derived by taking into account the singularity at the tip of the crack.

The majority of results obtained under this research program have already been documented either as interim reports published as AFOSR Technical Reports or in the form of technical papers published either in archival journals or in proceedings of technical conferences. The most recently obtained results of the following three subject matters have not been published:

(A) Creep and Viscoplastic Analysis by Assumed Stress Hybrid Finite Elements

(B) Three-Dimensional Crack Element by Assumed Stress Model

(C) Improvement of Plate and Shell Finite Elements by Hybrid Formulations

The first two items are presented respectively as Appendices (A) and (B) of this report. Each of these is made self-contained with individual numbering systems for equations, figures and references. The results of the last item has been written in a paper [P-20]* to be presented at the AIAA/ASME 18th Structures, Structural Dynamics and Materials Conference at San Diego, California, March 21-23, 1977.

In addition, the present research program has supported the works of one Ph.D. thesis and two M.S. theses in full one Ph.D. thesis in part and one Ph.D. thesis in progress. Also two students completed their M.S. theses and one is working on his Ph.D. thesis with some financial support for their computations under this research program.

Section 2 is to list all these publications, which have either already appeared or will soon be in print.

*Reference numbers headed by P refer to that in Section 2, List of Publications.

2. List of Publications under the Present Program

2.1 Technical Reports Published

1. Luk, Chih-Hung, "Assumed Stress Hybrid Finite Element Method for Fracture Mechanics and Elastic-plastic Analysis", AFOSR TR 73-0493, M.I.T. ASRL TR 170-1, December 1972, (also M.I.T., Department of Aeronautics and Astronautics, Ph.D. thesis).

2. Lasry, S.J., "Derivation of Crack Element-Stiffness Matrix by the Complex Variable Approach", AFOSR TR 73-1602, M.I.T. ASRL TR 170-2, February 1973, (also M.I.T., Department of Aeronautics and Astronautics, M.S. thesis).

3. Lee, Sung Won, "An Assumed Stress Hybrid Finite Element for Three-Dimensional Elastic Structural Analysis", AFOSR TR 75-0087, M.I.T. ASRL TR 170-3, May 1974, (also M.I.T., Department of Aeronautics and Astronautics, M.S. thesis).

4. Boland, P.L., "Large Deflection Analysis of Thin Elastic Structures by the Assumed Stress Hybrid Finite Element Method", AFOSR TR 76-1111, M.I.T. ASRL TR 170-4, October 1975, (also M.I.T., Department of Aeronautics and Astronautics, Ph.D. thesis).

2.2 Technical Papers Published

5. Pian, T.H.H., "Finite Element Methods by Variational Principles with Relaxed Continuity Requirement", Variational Method in Engineering, edited by C.A. Brebbia and H. Tottenham Southampton Uni. Press, 1972, pp. 3.1-3.24.

6. Tong, P. and Pian, T.H.H., "On the Convergence of Finite Element Method for Problems with Singularity", Int. J. Solids & Structures, Vol. 9, 1973, pp. 313-321.

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10. Tong, P., Mau, S.T., and Pian, T.H.H., "Deviation of Geometric Stiffness and Mass Matrices for Finite Element Hybrid Models", Int. J. Solids Structures, Vol. 10, 1974, pp. 919-932.

11. Pian, T.H.H., Tong, P, Luk, C.H. and Spilker, R.L., "Elastic-plastic Analysis by Assumed Stress Hybrid Model", Finite Element Methods in Engineering, edited by V.A. Pulmano and A.P. Kabaila, Univ. of New South Wales, Australia, 1974, pp. 419-434.

12. Pian, T.H.H., "Nonlinear Creep Analysis by Assumed Stress Finite Element Methods", AIAA J., Vol. 12, No. 12, December 1974, pp. 1756-1758.

13. Tong, P. and Pian, T.H.H.,"Application of Finite Element Method to Mixed-mode Fracture", Recent Advances in Engineering Science, Vol. 6, 1975, pp. 255-263.

14. Pian, T.H.H., Spilker, R.L. and Lee, S.W., "Elastic-plastic and Creep Analyses by Assumed Stress Finite Elements", Trans. 3rd, Int. Conf. on Structural Mechanics in Reactor Technology, Vol. 5, Part M, paper M 2/1, pp. 1-7.

15. Pian, T.H.H. and Lee, S.W., "Notes on Finite Elements for Nearly Incompressible Materials", AIAA J., Vol. 14, No. 6, June 1976, pp. 824-826.

16. Pian, T.H.H., "Hybrid Models for Three-Dimensional Crack Elements", Proceedings of a Workshop on Three-Dimensional Fracture Analysis, edited by L.E. Hulbert, Battell Columbus Laboratories, November 1976, pp. 84,85.

2.3 Technical Papers to be Published

17. Pian, T.H.H.,"Variational Principles for Incremental Finite Element Methods", to be published in J. Franklin Institute.

18. Boland, P.L. and Pian, T.H.H., "Large Deflection Analysis of Thin Elastic Structures by the Assumed Stress Hybrid Finite Element Method", Presented at 2nd National Symposium on

Computerized Structural Analysis and Design, Washington, D.C., March 29-31, 1976, (to be published in Computers and Structures).

19. Pian, T.H.H. and Boland, P.L., "Formulation of Large Deflection Shell Analysis by Assumed Stress Finite Element Method", presented at U.S.-German Symposium on Formulations and Computational Algorithms in Finite Element Analysis, M.I.T., August 9-13, 1976, (to be published by M.I.T. Press).

20. Lee. S.W., and Pian, T.H.H., "Improvement of Plate and Shell Finite Elements by Hybrid Formulations", AIAA paper No. 77-413. For presentation at AIAA/ASME 18th Structures, Structural Dynamics and Materials Conference, San Diego, CA, March 21-23, 1977.

2.4 Theses Supported in Part in the Present Program

21. Shimizu, T., "Axisymmetric Creep Analysis by Assumed Stress Hybrid Finite Element Method", M.I.T., Department of Aeronautics and Astronautics, M.S. thesis, September 1974.

22. Scharnhorst, Thomas, "Variational and Finite Element Formulations of Rubber-Like Materials", M.I.T., Department of Aeronautics and Astronautics, M.S. thesis, June 1976.

3. Summary of Research Findings

The research results obtained under this program are classified into the following four categories:

(A) Rationalization and extensions of variational formulations of finite element methods

(B) Development of crack elements

(C) Analysis of geometrically nonlinear problems

(D) Analysis of material nonlinear problems

The research findings under these categories are summarized in the following.

3.1 Rationalization and Extensions of Variational Formulations of Finite Element Methods

(A) In the application to finite element formulations in solid mechanics it has been demonstrated [P-5] that the conventional variational principles can be modified and extended by allowing the relaxation of the continuity requirements along the interelement boundary and that, correspondingly, many versions of hybrid models can be constructed. One example is that the so-called assumed stress hybrid elements can be derived using either a modified complementary energy principle [R-1, R-2]† or the Hellinger-Reissner principle if in the latter the assumed stresses are also in equilibrium [P-5].

†Reference numbers headed by R refer to that in the reference list at the end of this report.

The latter approach is computationally more efficient, particularly for the derivation of 3-D elements [P-3]. An eight-node isoparametric 3-D assumed stress element has been found to need a comparable amount of computing time for its generation in comparison with the conventional assumed displacement model, but to provide more accurate results. Another example is the utilization of a modified Hellinger-Reissner's principle to derive the geometric stiffness matrix for buckling analysis and the mass matrix for vibration analysis for assumed stress hybrid elements [P-10].

(B) Variational principles and modified principles in solid mechanics are extended to cover incremental finite element methods [P-17]. For those principles based on complementary energy, the assumed stresses need not satisfy the complete equilibrium conditions. Corrections for effects of equilibrium imbalance and compatibility mismatch at the beginning of each loading increment can be taken into account in a systematic manner.

(C) It has been found that hybrid formulations can be developed to reduce severe constraints that appear in the derivation of several finite elements by the conventional assumed displacement method, hence can lead to more satisfactory elements. In the assumed stress hybrid model, for example, the constraint for incompressibility is reduced to

only one condition, hence an ordinary hybrid stress element can be applied to problems which are very near to completely incompressible [P-15]. Two other examples are the use of hybrid elements to extend the range of applications for plate and shell problems [P-20].

3.2 Development of Crack Elements

A study has shown that the convergence rate of the finite element method for problems with singularities is very slow if ordinary elements are used [P-6]. Thus, it is desirable to employ special crack elements in which the proper singularities are taken into consideration. Two basic types of assumed stress hybrid models have been used to derive crack elements for plane elasticity problems. In the first one, the assumed stresses contain the correct distribution of the singular term but the additional terms consist of only polynomial expansions in the radial distance r from the crack tip [P-1]. In this case, several crack elements are needed around the crack tip. In the second case, the assumed stresses consist of complete series expansion of the exact solution around the crack tip. Thus, all terms satisfy both stress equilibrium and compatibility conditions and that contain not only the singular terms and polynomial expansion, but also terms involving $r^{n/2}$ with n being integers. In this case, the special element is one single element with an inbedded crack. [P-2, P-7, P-8, P-13].

Of these two types of crack elements, the second type is apparently more efficient both in the construction of the element and in the accuracy of the solution. Both elements have been demonstrated to be superior among the existing finite element methods for the determination of stress intensity factors [R-3]. The second approach for hybrid crack elements has been extended under other ASRL research programs sponsored by the Air Force, to problems involving bi-material interface [R-4, R-5]. A nine-node hybrid crack element with a code name of PCRK 59 has been employed to compute stress intensity factors for several aircraft structural details including attachment lugs, centered and offset fastener holes [R-6, R-7, R-8, R-9]. This element is now incorporated in the library of subroutines of the FEABL-2 software [R-10] and has also been incorporated by several aerospace manufacturers in their structural analysis computing programs.

The first approach for hybrid crack elements has been extended to 3-D crack analysis [P-16]. A brief summary of this development is given in Appendix B of this report.

3.3 Analysis of Geometrically Nonlinear Problems

(A) The bifurcation and the postbuckling behaviors of shell of revolution have been examined by the finite element method [P-9]. The symmetric prebuckling deformation of the shell is first formulated and the stiffness matrices of

various harmonics are examined for the possibility of bifurcation. A postbuckling analysis is then performed to examine the stability of such bifurcated mode. The effect of symmetric and asymmetric imperfections and of material orthotropy on the buckling load of spherical shells has been studied.

(B) The assumed stress hybrid finite element model has been demonstrated to be suitable for incremental solution of thin elastic structures in large deflections. An inconsistent model may be adopted in place of a consistent one which would involve a much more complicated task of assuming stresses to satisfy the entire equilibrium equations. The discrepancy, however, can be taken into account by the inclusion of correction terms corresponding to checks on stress equilibrium and compatibility at the beginning of each loading increment. The method has been applied to plate and shells in large deflections but in small strains [P-4, P-18, P-19] and to rubber-like material in the finite strain range (P-22].

3.4 Analysis of Material Nonlinear Problems

(A) For elastic-plastic analyses, it has been shown that both the tangent stiffness approach and the initial stress approach can be adapted in conjunction with the assumed stress hybrid model [P-1, P-11, P-14]. Results have demonstrated that the hybrid elements are more efficient than the conventional assumed displacement element from the point of view of solution accuracy and computing effort.

(B) The assumed stress hybrid elements have also been applied to nonlinear creep analysis (P-12, P-14, P-21]. The resulting equations are in the form of a system of nonlinear first order deferential equations for which many standard numerical techniques are available. Appendix A presents some results and also demonstrates that the elastic-plastic problems can be solved by the viscoplasticity approach, hence can also be solved by the present creep analysis program.

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10. Orringer, O. and French, S.E., "FEABL User's Guide", AFOSR-TR-72-2228, M.I.T. ASRL TR 162-3, August 1972.

APPENDIX A

Creep and Viscoplastic Analyses by Assumed Stress Hybrid Finite Elements

T. H. H. Pian and S. W. Lee

ABSTRACT

A hybrid stress finite element formulation is derived for creep analysis and elastoplastic analysis by viscoplastic formulation. Application of the Hellinger-Reissner principle leads to a system of nonlinear first order ordinary differential equations with stress parameters in each element as variables. Examples involving four-node plane stress element and solid of revolution element indicate that, for elements with a low order displacement field, the assumed stress hybrid formulation is computationally more efficient than the conventional assumed displacement formulation.

I. Introduction

Creep and viscoplastic problems are nonlinear in nature and in general require numerical solution methods. The finite element method based on the assumed displacement models has been applied to this type of problem treating creep or viscoplastic strains as initial strains. Percy et al [1],

Greenbaum and Rubinstein [2], Sutherland [3], Chang and Rashid [4], Branca and Boresi [5], Donea and Giuliani [6], etc. solved small displacement creep problems while Cyr and Teter [7] solved large deformation creep problems combined with plastic effect. Numerical stability is very important for all numerical analyses and sufficiently small time steps were used in these works. Cormeau [8] developed a formulation for viscoplastic analysis of solids utilizing the constitutive law proposed by Perzyna [9] and applied it to elastoplastic analysis and to analysis of transient creep under the time hardening law. A theoretical stability criterion has been established in his formulation. Bodner et al [10,11] proposed an alternative viscoplastic constitutive law and applied it to the finite element method.

In this work an assumed stress hybrid finite element method for small deformation creep analyses will be derived from the Hellinger-Reissner principle with initial strains. The resulting system of nonlinear first order ordinary differential equations can be solved by various Runge-Kutta methods. Finally by employing Perzyna's constitutive law for viscoplasticity, the timewise solution procedure is to be used for solution of elastoplastic problems.

2. Constitutive Equations

(a) Creep Law

For the present purpose, phenomenological description of creep will be adopted. In the study of long duration creep the steady state creep law is used and the corresponding uniaxial creep strain rate is expressed by

$$\dot{\varepsilon}^{c} = F(\sigma, T) \tag{1}$$

where σ is the uniaxial stress and T is the temperature.

If the effects of σ and T are assumed to be separable, then

$$\dot{\varepsilon}^{c} = f(\sigma)\, g(T) \tag{2}$$

For the stress-dependent function, we use Norton's power law, i.e.,

$$f(\sigma) = B\sigma^{n} \tag{3}$$

where B and n are material constants. For the present study it will be assumed from now on that the temperature remains constant. In the case of transient creep, the time hardening law and the strain hardening law are available. Under constant

temperature and constant stress the uniaxial creep strain may be expressed as

$$\varepsilon = A\sigma^{m} t^{k} \tag{4}$$

where A, m, k are material constants and t represents time. Differentiating eq. (4) with respect to time leads to

$$\dot{\varepsilon}^{c} = Ak\sigma^{m} t^{k-1} \tag{5}$$

The variable t can be eliminated from eq. (5) by solving eq. (4) for t and substituting into eq. (5). The result is

$$\dot{\varepsilon}^{c} = k\sigma^{\frac{m}{k}} A^{\frac{1}{k}} (\varepsilon^{c})^{1-\frac{1}{k}} \tag{6}$$

Equations (5) and (6) are called the time hardening law and the strain hardening law respectively. The time hardening law can be written in a form which resembles the steady state creep law by defining a parameter $\tau = t^{k}$ and differentiating eq. (4) with respect to τ to yield

$$\dot{\varepsilon}^{c} = A\sigma^{m} \tag{7}$$

Under multiaxial stress condition creep strain rates are expressed as

$$\dot{\varepsilon}_{ij}^{\,c} = \dot{\varepsilon}_e^{\,c} \frac{\partial F}{\partial \sigma_{ij}} \tag{8}$$

where $\dot{\varepsilon}_e^c$ is the equivalent creep strain rate and the yield condition is represented by

$$F(\sigma_{ij}) = 0 \tag{9}$$

By substituting the uniaxial creep strain rates of eqs. (6) and (7) into eq. (8) for the equivalent creep strain rate and using the equivalent stress σ_e in place of the uniaxial stress, we obtain

$$\dot{\varepsilon}_{ij}^{\,c} = A\sigma_e^{\,m} \frac{\partial F}{\partial \sigma_{ij}} \tag{10}$$

for the steady state creep or creep under the time hardening law and

$$\dot{\varepsilon}_{ij}^{\,c} = k\sigma_e^{\frac{m}{k}} A^{\frac{1}{k}} (\varepsilon_e^c)^{1-\frac{1}{k}} \frac{\partial F}{\partial \sigma_{ij}} \tag{11}$$

for creep under the strain hardening law. In eq. (10) the creep strain rate is determined by the present stress values while in eq. (11) it is expressed in terms of the present stresses and the equivalent creep strain or creep strains.

If the Mises-Hencky yield criterion is used, then

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{3}{2} \frac{S_{ij}}{\sigma_e} \tag{12}$$

where S_{ij} are the deviatoric stresses.

(b) Perzyna's Viscoplastic Law

For materials undergoing viscoplastic process, multi-axial viscoplastic strain rates are expressed as follows

$$\dot{\varepsilon}_{ij}^{p} = \gamma \phi \frac{\partial F}{\partial \sigma_{ij}} \tag{13}$$

where F is the same yield function as in eq. (9) with uni-axial yield stress σ_Y and

$$\begin{aligned} \phi &= \sigma_e - \sigma_Y \quad && \text{if} \quad \sigma_e \geq \sigma_Y \\ &= 0 && \text{if} \quad \sigma_e < \sigma_Y \end{aligned} \tag{14}$$

and γ is a material constant. If the yield stress is constant, the viscoplastic strain rates are determined by stresses only. The above representation has been proposed by Perzyna [9]. It is different from the classical inviscid plasticity theory in that it permits the stresses to exceed the yield stress. The visco-plastic formulation can be used to solve classical plasticity problems. This is achieved by

finding a steady state solution under a given load increment.

(c) Bodner-Partom's Law

In Bodner-Partom's constitutive law inelastic strain rates are expressed as follows

$$\dot{\varepsilon}_{ij}{}^{P} = f(J_2)\, S_{ij}/J_2 \tag{15}$$

where J_2 is the second invariant of the deviatoric stresses and the function f include material constants and plastic work done. No yield stress is defined and thus inelastic strain components are present all the time from the moment a load is applied. In the present investigation we will not use this constitutive law. But it can be treated in the same manner as creep laws or Perzyna's law.

3. Finite Element Formulation

An assumed stress hybrid finite element can be formulated by a modified complementary energy principle or by the Hellinger-Reissner principle. For the example problems carried out in this investigation, the Hellinger-Reissner principle is suitable since it is easy to construct compatible displacement fields. For creep and viscoplastic analyses, the Hellinger-Reissner principle is extended to include the

initial strains. The functional π_R is written in terms of stress and displacement rates as follows.

$$\Pi_R = \sum_n \int_{V_n} \Big[\dot{\sigma}_{ij} \tfrac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) - \tfrac{1}{2} S_{ijk\ell} \dot{\sigma}_{ij} \dot{\sigma}_{k\ell} - \dot{\sigma}_{ij} \dot{\varepsilon}^{\circ}_{ij} - \dot{\bar{F}}_i \dot{u}_i \Big] dv - \int_{S_{\sigma_n}} \dot{\bar{T}}_i \dot{u}_i \, dS \tag{16}$$

where

$\dot{\sigma}_{ij}$ = stress rate tensor

$\dot{u}_i$ = displacement rates

$S_{ijk\ell}$ = elastic coefficient tensor

$\dot{\varepsilon}^{\circ}_{ij}$ = initial strain rates, i.e., creep strain or viscoplastic strain rates in the present case

$\dot{\bar{F}}_i$ = applied body force rates per unit volume

$\dot{\bar{T}}_i$ = applied traction rates

V_n = volume of the element

S_{σ_n} = surface of the nth element over which tractions are applied

In the matrix form this functional can be written as

$$\Pi_R = \sum_n \int_{V_n} \Big[\dot{\underset{\sim}{\sigma}}^T (\underset{\sim}{D} \dot{\underset{\sim}{u}}) - \tfrac{1}{2} \dot{\underset{\sim}{\sigma}}^T \underset{\sim}{S} \dot{\underset{\sim}{\sigma}} - \dot{\underset{\sim}{\sigma}}^T \dot{\underset{\sim}{\varepsilon}}^{\circ} - \dot{\bar{\underset{\sim}{F}}}^T \dot{\underset{\sim}{u}} \Big] dv - \int_{S_{\sigma_n}} \dot{\bar{\underset{\sim}{T}}}^T \dot{\underset{\sim}{u}} \, dS \tag{17}$$

The above expression is valid for small deformation. The corresponding variational principle for large deformation has been given by Sanders et al [12] and Pian [13]. In

eq. (16) the displacements and the stresses are independent quantities. For the hybrid stress finite element implementation, stress and dispalcement rates are assumed in terms of unknown stress rate parameters $\dot{\tilde{\beta}}$ and nodal dispalcement rates $\dot{\tilde{q}}$ respectively, i.e.,

$$\dot{\tilde{\sigma}} = \tilde{P}\,\dot{\tilde{\beta}} + \tilde{P}_F\,\dot{\tilde{\beta}}_F$$

$$\dot{\tilde{u}} = \tilde{A}\,\dot{\tilde{q}} \tag{18}$$

then

$$(\tilde{D}\dot{\tilde{u}}) = \tilde{B}\,\dot{\tilde{q}}$$

Here the assumed stress rates are made to satisfy the equilibrium condition. Thus $\tilde{P}\,\dot{\tilde{\beta}}$ is the homogeneous solution of the equilibrium equations and $\tilde{P}_F\,\dot{\tilde{\beta}}_F$ is a particular solution of the equilibrium equations. Substituting eq. (18) into eq. (17) and performing necessary integrations, we obtain

$$\pi_R = \sum_m \Big(\dot{\tilde{\beta}}^T \tilde{G}\,\dot{\tilde{q}} + \dot{\tilde{\beta}}_F^T \tilde{G}_F\,\dot{\tilde{q}} - \frac{1}{2}\dot{\tilde{\beta}}^T \tilde{H}\,\dot{\tilde{\beta}}$$

$$- \dot{\tilde{\beta}}^T \tilde{H}_F\,\dot{\tilde{\beta}}_F - \dot{\tilde{\beta}}^T \dot{\tilde{G}}_o - \dot{\tilde{q}}^T \dot{\tilde{Q}}_A \Big) \tag{19}$$

where

$$\tilde{G} = \int_{V_m} \tilde{P}^T \tilde{B}\, dv \qquad \tilde{G}_F = \int_{V_m} \tilde{P}_F^T \tilde{B}\, dv$$

$$\dot{\tilde{G}}_o = \int_{V_m} \tilde{P}^T \dot{\tilde{\varepsilon}}^o\, dv \qquad \tilde{H}_F = \int_{V_m} \tilde{P}^T \tilde{S}\, \tilde{P}_F\, dv$$

$$\tilde{H} = \int_{V_m} \tilde{P}^T \tilde{S}\, \tilde{P}\, dv \qquad \dot{\tilde{q}}^T \dot{\tilde{Q}}_A = \int_{S_{\sigma_m}} \dot{\tilde{T}}^T \dot{\tilde{u}}\, dS \tag{20}$$

Taking $\delta\pi_R = 0$ with respect to $\dot{\tilde{\beta}}$ gives

$$\tilde{G}\dot{\tilde{q}} - \tilde{H}\dot{\tilde{\beta}} - \tilde{H}_F\dot{\tilde{\beta}}_F - \dot{\tilde{G}}_o = 0$$

or

$$\dot{\tilde{\beta}} = \tilde{H}^{-1}(\tilde{G}\dot{\tilde{q}} - \tilde{H}_F\dot{\tilde{\beta}}_F - \dot{\tilde{G}}_o) \tag{21}$$

Substituting eq. (21) into eq. (19) yields

$$\Pi_R = \sum_n \left[\frac{1}{2}\dot{\tilde{q}}^T \tilde{k}_n \dot{\tilde{q}} - \dot{\tilde{q}}^T(\dot{\tilde{Q}}_n + \dot{\tilde{Q}}_n^o) \right] \tag{22}$$

where

$$\begin{aligned} \tilde{k}_n &= \tilde{G}^T\tilde{H}^{-1}\tilde{G} = \text{element stiffness matrix} \\ \dot{\tilde{Q}}_n &= \dot{\tilde{Q}}_A + (\tilde{G}^T\tilde{H}^{-1}\tilde{H}_F - \tilde{G}_F^T)\dot{\tilde{\beta}}_F \\ &= \text{rate of applied nodal load} \\ \dot{\tilde{Q}}_n^o &= \tilde{G}^T\tilde{H}^{-1}\dot{\tilde{G}}_0 = \text{rate of equivalent nodal load due to} \\ &\qquad \text{inelastic strain} \end{aligned} \tag{23}$$

The element stiffness matrix and nodal load can be assembled to yield

$$\Pi_R = \frac{1}{2}\dot{\tilde{q}}^T \tilde{K}\dot{\tilde{q}} - \dot{\tilde{q}}^T(\dot{\tilde{Q}} + \dot{\tilde{Q}}^o) \tag{24}$$

and the stationariry condition of π_R with respect to $\dot{\tilde{q}}$ gives

$$\tilde{K}\dot{\tilde{q}} = \dot{\tilde{Q}} + \dot{\tilde{Q}}^o \tag{25}$$

Solving eq. (25) for $\dot{q}$ and substituting into eq. (21), we obtain

$$\dot{\beta} = H^{-1}(G K^{-1} G^{T} H^{-1} - I)\dot{q}_{o} - H^{-1} G K^{-1} \dot{Q} - H^{-1} H_{F} \dot{\beta}_{F} \tag{26}$$

For problems involving state creep, creep under the time hardening law and viscoplasitc law with constant yield stress.

$$\dot{\varepsilon}^{\circ} = f(\sigma) = f(\beta) \tag{27}$$

and thus from eq. (26)

$$\dot{\beta} = g(\beta) \tag{28}$$

On the other hand, for creep under the strain hardening law or for viscoplasticity law with yield stress dependent on the viscoplastic strains

$$\begin{aligned} \dot{\beta} &= g_{1}(\beta, \varepsilon^{\circ}) \\ \dot{\varepsilon}^{\circ} &= g_{2}(\beta, \varepsilon^{\circ}) \end{aligned} \tag{29}$$

Equations (28) and (29) are systems of nonlinear first order ordinary differential equations with $\underset{\sim}{\beta}$ and $\underset{\sim}{\beta}, \underset{\sim}{\varepsilon}^{o}$ as unknowns respectively. In the finite element formulation, integrations over an element are performed numerically. Thus, in eq. (29) the vector $\dot{\underset{\sim}{\varepsilon}}^{o}$ represents inelastic strains at Gaussian integrations points. The assumed stress hybrid formulation for the steady state creep and for transient creep under the time hardening law has been given by Pian [14].

In the conventional assumed displacement method based on the principle of virtual work with initial strains, the unknowns are stresses $\underset{\sim}{\sigma}$ at Gaussian integration points instead of $\underset{\sim}{\beta}$ and we have the same type of equations as eqs. (28) and (29) with $\underset{\sim}{\beta}$ replaced by $\underset{\sim}{\sigma}$. The amount of computing time depends on the number of unknowns. For elements with a low order assumed displacement field, assumed stress hybrid formulation will result in a problem with fewer number of unknowns than that by the corresponding assumed displacement method. For example, consider a four node plane stress element derived by 2 x 2 integration rule. An element based on the displacement approach will have three stress components at each Gaussian point and thus has twelve unknown stresses per element. On the other hand a hybrid stress element with 7 β's has seven unknowns. Similarly, for the assumed

stress hybrid model, a four-node axisymmetric element with 8 β's and a eight-node three dimensional element with 24 β's have fewer number of unknowns that the corresponding elements by the assumed displacement model. This advantage of assumed stress hybrid elements disappear as the number of nodes are increased for an element since more β's are needed to prevent hazardous kinematic modes. For example for an eight node plane stress element the number of unknown β's in the hybrid formulation and the number of σ's for the displacement formulation are comparable to each other.

4. Solution Method

Equations (28) and (29) can be solved timewise by numerical methods such as Runge-Kutta methods [15,16] using elastic solution as initial conditions. All these methods have numerical stability limit. For the integration of eq. (28) stability criterion developed by Cormeau [8] has been utilized. The well known fourth order explicit Runge-Kutta method was used for steady state creep and creep under the time hardening law. In the case of the time hardening law, the integration is performed for the time parameter $\tau = t^k$. The real time is recovered by $t = \tau^{1/k}$. On the other hand the midpoint Runge-Kutta method is considered more appropriate for elastoplastic problems where a lower order method was chosen due to the possibility of piecewise continuity of stresses.

To solve elastoplastic problems, it is assumed at first that the present state is on the yield surface or in an elastic state. Given a load increment, the material will flow according to eq. (13) if $\sigma_e > \sigma_Y$. But if the structure is stable, the plastic flow will stop after a certain time and a steady state is reached under the given load. This steady state will be the one which is obtainable by the classical plasticity formulation. In actual calculations it is assumed that a steady state has been reached if

$$\frac{\sigma_e - \sigma_Y}{\sigma_Y} \leq \varepsilon \tag{30}$$

where ε is an arbitrarily assigned value. For the present numerical solution it is chosen as 0.01.

For creep problems under the strain hardening law, the time increment for the numerical integration of eq. (29) has been determined such that

$$\dot{\varepsilon}_e^c \; \Delta t < \text{total strain at time } t \times a \tag{31}$$

where a is a preassigned value. The value of a = 0.05 was used in the present work. It is to be noted that for materials with k < 1, eq. (11) and hence eq. (29) are singular at time t = 0 since $\varepsilon_e^c = 0$. Therefore it is necessary to use a method which does not require evaluation of the creep strain

rate at t = 0. A fourth order implicit Runge-Kutta method developed by Butcher [17] was chosen for this purpose. After one time increment, it is switched to the fourth order explicit method since the explicit method requires less computing time for the same accuracy.

5. Numerical Examples

Two assumed stress hybrid elements are used for numerical solution of example problems. They are a four-node quadrilateral plane stress element with the following stress assumption

$$
\begin{aligned}
\sigma_x &= \beta_1 + \beta_4 x + \beta_5 y \\
\sigma_y &= \beta_2 + \beta_6 y + \beta_7 x \\
\sigma_{xy} &= \beta_3 - \beta_4 y - \beta_6 x
\end{aligned}
\tag{32}
$$

and a four-node quadrilateral axisymmetric element [19] with the following assumed stresses

$$
\begin{aligned}
\sigma_r &= \beta_1 + \beta_2/r + \beta_3 z/r \\
\sigma_\theta &= \beta_4 \\
\sigma_z &= \beta_5 + \beta_6/r - \beta_7 z/r \\
\sigma_{rz} &= -\beta_1 z/r + \beta_4 z/r + \beta_7 + \beta_8/r
\end{aligned}
\tag{33}
$$

The isoparametric representation is used for displacement assumption. Among the following examples, the first two

problems deal with creep and the last one with elastic-plastic analysis. Computations were performed with an IBM 370/165 machine at the M.I.T. Information Processing Center.

(a) Creep of a thick cylinder under internal pressure.

An infinitely long thick cylinder subjected to internal pressure was solved as a plane strain problem using six axisymmetric elements. The geometry, load and material properties are given in fig. 1. Results are given in figs. 1 and 2. The time hardening law and the strain hardening law gave almost identical results. All results are close to those obtained by Greenbaum and Rubinstein [2].

(b) Creep of a rotating disk

This is a problem involving distributed body forces. The geometry and material constants are given in fig. 3. First it was analyzed by a row of six axisymmetric elements and then by a row of twelve plane stress elements. In the plane stress element a particular solution of the equilibrium equations was chosen for the averages of the body forces over the volume of the element. For the axisymmetric element the particular solutions were simply set to zero. Discussions relating these two approaches are given in ref. 18. Calculated results for the time hardening law are shown in figs. 3 and 4. In fig. 3, t and m represent the thickness and the density of the disk respectively.

(c) An infinitely long thick cylinder of elastic-perfectly plastic material under internal pressure p

The geometry and material properties are as follows.

Inner radius a = 5 inch.

Outer radius b = 10 inch.

Young's modulus E = 10^7 psi

Poisson's ration ν = 0.3

Yield stress σ_y = 20000 psi

Table 1 shows the radial displacement at the inner wall calculated by both the assumed stress hybrid method and the assumed displacement method using eight axisymmetric elements. The 2 x 2 Gaussian integration points were used for numerical integration. The exact solutions were obtained according to the procedure described in Ref. [21] based on the classical plasticity theory. The accuracy of the finite element solution decreases as the load approaches collapse value and more fictitious time steps are needed to reach a steady state under the given load increment. But it is seen that the assumed stress hybrid method gives better solution in less computing time than the assumed displacement method.

6. Conclusion

An assumed stress hybrid finite element formulation has been derived for viscoplastic and creep analyses. The practicality of the method has been demonstrated by example

problems. For elements with a low order of assumed displacement field the assumed stress hybrid formulation is computationally more efficient than the conventional displacement method for small deformation problems.

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Table 1. Radial Displacement x 10 at the Inner Wall

p/σ_y	Exact Solution	Hybrid Method	Displacement Method
0.5117	0.1002	0.1006	0.0995
0.5786	0.1206	0.1210	0.1191
0.6345	0.1435	0.1432	0.1416
0.6806	0.1689	0.1688	0.1665
0.7179	0.1966	0.1965	0.1923
0.7472	0.2264	0.2255	0.2191
0.7692	0.2583	0.2550	0.2474
0.7846	0.2922	0.2871	0.2746
0.7937	0.3278	0.3176	0.2963
0.7972 (collapse)	0.3650	0.3328	0.3061
CPU Time	(sec.)	6.30	9.29

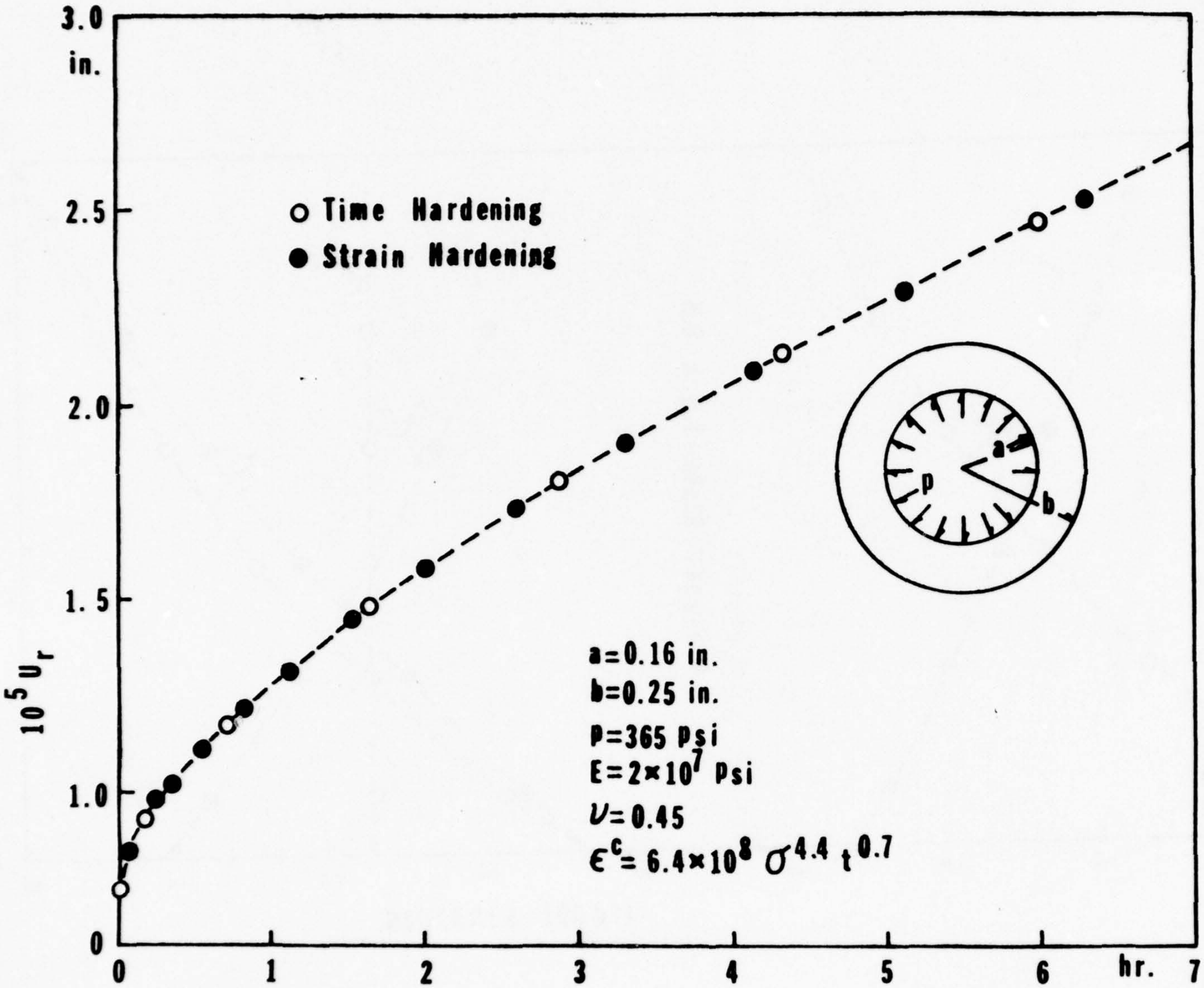

FIG.1 RADIAL DISPLACEMENT AT THE INNER WALL VS. TIME

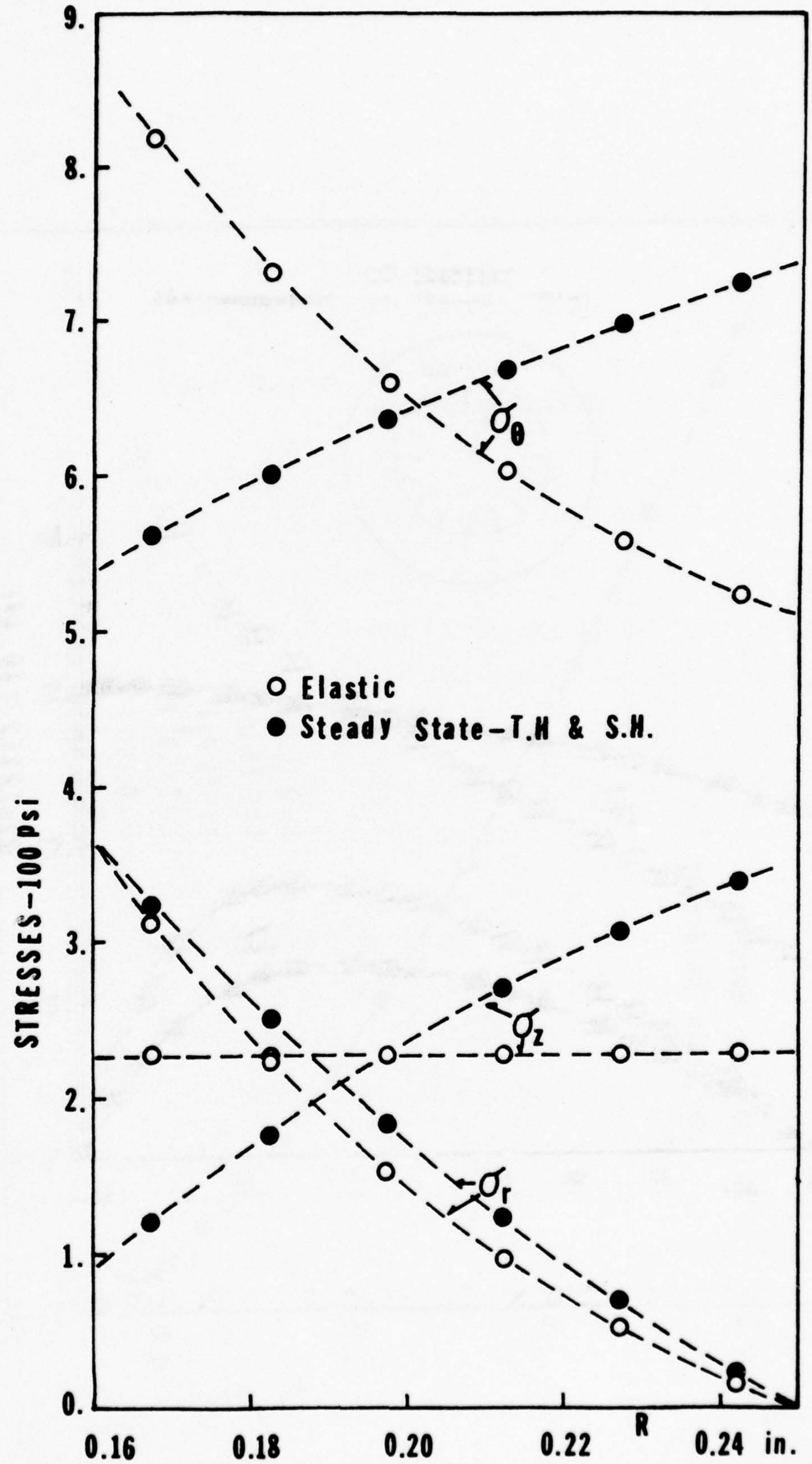

FIG. 2 STRESS VS. RADIUS

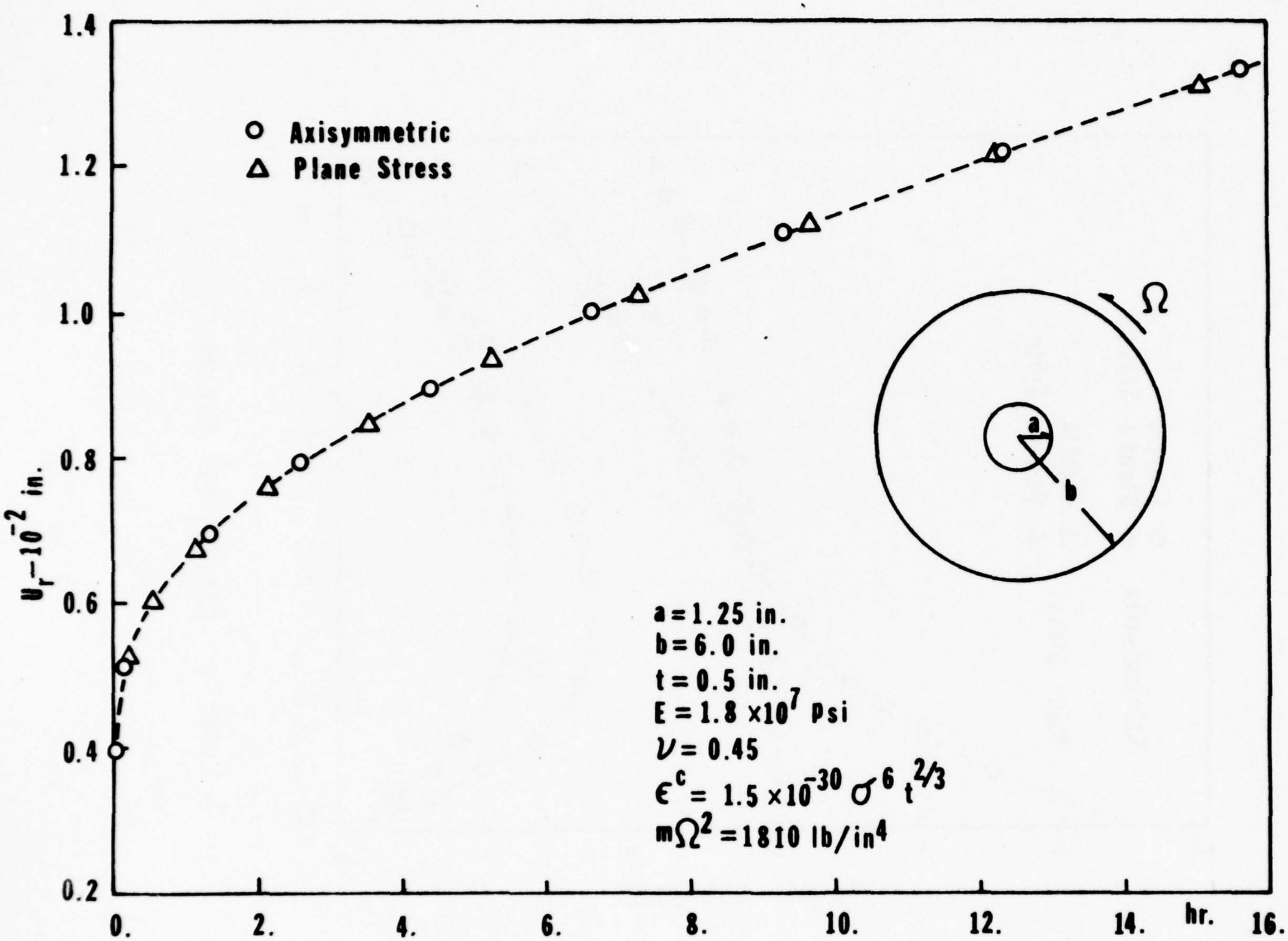

FIG. 3 RADIAL DISPLACEMENT AT R = 1.25 in. VS. TIME

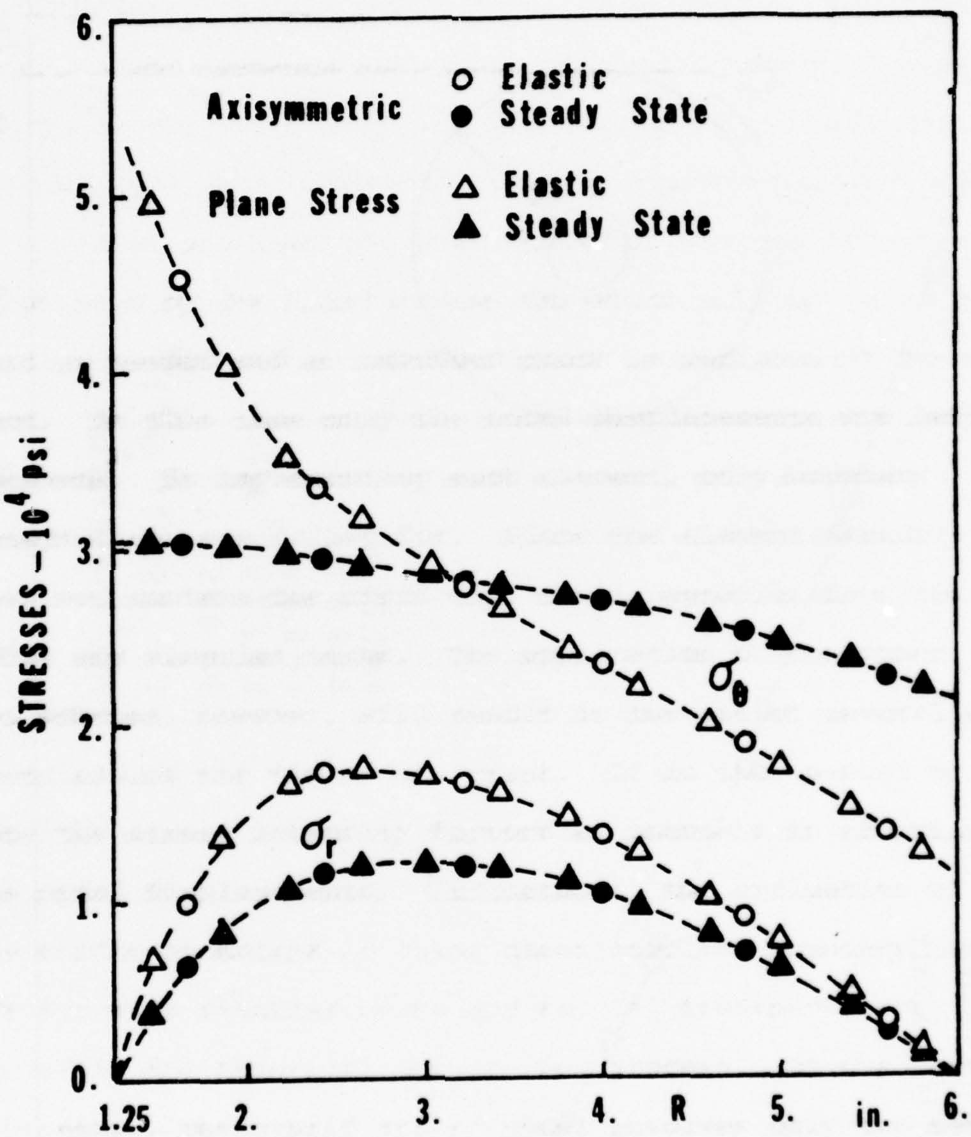

FIG. 4 STRESS VS. RADIUS

APPENDIX B

Three-Dimensional Crack Element by Assumed Stress Hybrid Model

T. H. H. Pian and K. Moriya

1. Introduction

There exist three basic finite element hybrid models for the development of special singular elements for linear fracture mechanics [1]. The basic schemes for these models are as follows.

(1) A scheme which is based on the assumed equilibrating stress field which contains the singular stress term and also satisfies the compatibility condition inside the element, and on independently assumed boundary displacements. Such model may be interpreted as either a hybrid stress model or a hybrid displacement model.

(2) A scheme which is based on assumed equilibrating stress field including the singular term and on independently assumed boundary displacements. Such model is a hybrid stress model.

(3) A scheme which is based on assumed displacement field including the singular term and on independently assumed

boundary displacements and boundary tractions. Such a model is a hybrid displacement model.

The applications of these three schemes to two dimensional crack elements have been reported by Tong, Pian and Lasry [2], by Pian, Tong and Luk [3], and by Atluri, Kobayashi and Nakagaki [4,5] respectively. It appears that for two-dimensional problems the most desirable element is one formulated by the first scheme for which only one crack element is needed and an imbedded crack is included in the element. In this case only the nodal displacements are left as unknowns. In implementing such element, only boundary integrations are called for. Since the element boundary does not contain the crack tip, the integration does not involve any singular terms. The application of the other two schemes, however, will result in the use of several elements around the tip of the crack. It is then needed to introduce the stress intensity factors as unknowns in addition to the nodal displacements. Furthermore, the evaluation of the stiffness matrix in these cases involves boundary integration with singular terms and special treatments are needed in the numerical integration process. Of the last two approaches, the hybrid stress model involves only two sets of field variables, while the hybrid displacement model contains three sets of field variables.

For three-dimensional fracture mechanics, it is unfortunate that the first approach is no longer applicable. First of all the only available near field solution near a crack front is a two-dimensional behavior in terms of r and θ, where, as shown in fig. 1, r is the radial distance from the crack front, and θ is the angle from the n-t plane. Here the t-axis is tangent to the crack front, the n-axis is on the plane of the crack and z is normal to the crack plane.
The singular behavior of the stress field and the corresponding displacement distribution are given by the following equations.

$$\begin{Bmatrix} \sigma_n \\ \sigma_z \\ \sigma_t \\ \tau_{nz} \\ \tau_{zt} \\ \tau_{nt} \end{Bmatrix} = \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} K_1 \begin{Bmatrix} \cos\frac{\theta}{2}(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}) \\ \cos\frac{\theta}{2}(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}) \\ 2\nu\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \end{Bmatrix} + K_2 \begin{Bmatrix} -\sin\frac{\theta}{2}(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}) \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \\ -2\nu\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}) \end{Bmatrix} \\ K_3 \begin{Bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{Bmatrix} \end{Bmatrix}$$

$$\begin{Bmatrix} u_n \\ u_z \\ u_t \end{Bmatrix} = \frac{1}{G}\sqrt{\frac{2r}{\pi}} \begin{Bmatrix} \frac{K_1}{8} \begin{Bmatrix} (5 - 8\nu)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \\ (7 - 8\nu)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2} \end{Bmatrix} + \frac{K_2}{8} \begin{Bmatrix} (9 - 8\nu)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \\ (-3 + 8\nu)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \end{Bmatrix} \\ K_3\sin\frac{\theta}{2} \end{Bmatrix} \tag{1}$$

Here, K_1, K_2 and K_3 are the stress intensity factors and are of unknown distributions along the crack front. Thus, a complete series solution with each term as known function of r, θ and t is not readily available. Also, since the crack front will always intersect the boundary of the element, it is necessary in the formulation of the element stiffness matrix to evaluate some surface integrals that are singular. Atluri et al have applied the third approach, or the hybrid displacement model to derive three-dimensional crack elements. The present note is to describe the development of three-dimensional crack elements using the second approach or the hybrid stress model.

2. Three-Dimensional Crack Elements by Assumed Stress Hybrid Model

For the presently derived three-dimensional elements the edges are all made straight lines. Thus, if in a problem the crack front is curved it must be approximated by an assemblage of straight segments. Each segment of the crack front, then, serves as the common edge of a group of brick type special crack front elements. An example arrangement with four such elements is shown in Fig. 2. These elements are classified as type A element which contains the crack surface and type B element which does not contain the crack surface.

elements labelled as A' and B' are those with the crack front located at the upper corners of the block elements. In analyzing a complete solid with general mesh pattern, regular finite elements are used to surround these special elements.

The variational functional π to be used in the derivation of this element is written as

$$\Pi(\sigma_{ij}, \tilde{u}_i) = \sum_{n=1}^{N}\{\int_{V_n} \frac{1}{2} C_{ijkl}\, \sigma_{ij}\, \sigma_{kl}\, dV - \int_{\partial V_n} T_i\, \tilde{u}_i\, dS + \int_{S_{\sigma_n}} \tilde{T}_i\, \tilde{u}_i dS\} \quad (2)$$

where the stress σ_{ij} satisfies equations of equilibrium, $C_{ijk\ell}$ is the elastic compliance tensor, T_i is the surface traction obtained from the stresses σ_{ij}, $\tilde{u}_i$ is the boundary displacement and $\tilde{T}_i$ is prescribed surface traction over S_{σ_n}. Thus, the independent field variables are the equilibrating stresses σ_{ij} and boundary displacements $\tilde{u}_i$. In a crack front element, the assumed stress includes not only the regular polynomial terms but also the asymptotically exact term which is derived from the well known local solution of an embedded elliptical crack, and is given in eq. (1). Hence, the assumed stress in the crack fron element is written as

$$\sigma_{ij} = P_{s_{im}}(1/\sqrt{r}, \theta)\, K_m + P_{r_{in}}(x, y, z)\, \beta_n \quad (3)$$

The first term of the above equation corresponds to the singular solution eq. (1), while the second term contains only regular polynomials. Both terms satisfy homogeneous equilibrium equations and the stress free conditions over the crack surfaces.

The assumed boundary displacement $\tilde{u}_i$ which is expressed in terms of nodal displacement contains asymptotically correct variation of displacement on the faces that meet or intersect the crack front line, and the condition of displacement compatibility is completely satisfied across all interelement boundaries. Thus, $\tilde{u}_i$ is expressed as

$$\tilde{u}_i = L_{ij} q_j \tag{4}$$

where q_j is a generalized (nodal) displacement and L_{ij} is an interpolation function. Inserting eqs. (3) and (4) into eq. (2) and eliminating stress parameters β_i by using stationary condition for the functional π with respect to $\delta\beta_i$, π can be expressed in terms of generalized displacements q_i and the stress intensity factor K_i. For the entire structure, terms associated with the same nodal displacement can be assembled. Again, the stationary condition of $\pi(q_i, K_i)$ with respect to δq_i and δK_i leads to the final algebraic equation of the form,

$$\begin{bmatrix} [K_{rr}] & [K_{rs}] \\ [K_{rs}]^T & [K_{ss}] \end{bmatrix} \begin{Bmatrix} \{q_i\} \\ \{K_i\} \end{Bmatrix} = \begin{Bmatrix} \{Q_i\} \\ \{0\} \end{Bmatrix} \tag{5}$$

Solving the above equations, the generalized displacements q_i and the stress intensity factors K_i can be obtained. It is also possible to consider only the group of elements at the crack front and form a superelement. The equations are of the same form as eq. (5). By eliminating $\{K_i\}$, a stiffness matrix [K] of the superelement given by

$$[K] = [K_{rr}] - [K_{rs}][K_{ss}]^{-1}[K_{rs}]^T \tag{6}$$

can be obtained. Such stiffness matrix can be used in the ordinary matrix displacement method.

Three types of singular element, i.e., 8-node, 12-node and 16-node elements are constructed (fig. 3). The 16-node element has mid-side nodes on all edges on the face that intersects the crack front, whereas the 12-node element has mid-side nodes only on edges which are not connected with the crack front. Table 1 lists the number of β's used in the various elements. Figure 3 also indicates typical assemblages of crack elements. The 12-node, 20-node and 26-node half

elements are for problems which are symmetric about the plane of the crack, while the 20-node, 36-node and 46-node superelements are for general asymmetric problems.

To assess the accuracy of these special crack front elements, a simple test has been performed. Element nodal displacements are calculated from the known analytical solution that corresponds to $K_1 = K_2 = K_3 = 1.00$. Using these displacements as input data, K_1, K_2 and K_3 are then computed from the numerically generated element matrices. The results obtained by using different types of singular element and by using different half elements and superelements are listed in Table 1. It is seen that they all compare favorably with the correct values. Obviously elements with more nodes yield more accurate solutions.

The elements have been utilized to analyze several fracture test specimens of typical geometries, i.e., a center crack specimen, a single edge crack specimen and a double edge crack specimen under uniform tension along the faces parallel to the crack plane. The geometries of these specimens and mesh subdivision used are shown in fig. (4) and (5) respectively. By the double or triple symmetry of geometry, only one quarter or one eighth of these specimens are considered. Figures (6), (7) and (8) show the variation of the

value of the calculated stress intensity factor K_1 across the plate thickness. Another example problem that has been analyzed with this special element is a compact tension specimen shown in fig. (9). Because of the double symmetry of geometry only one quarter of the specimen is needed in the analysis. Figure (10) shows the mesh pattern used for this problem. There are 168 nodes, 85 elements and 504 degrees of freedom. It is seen that the pin hole is removed in this model and the loading is represented by a distributed line load normal to the crack surface.

Figures (11) and (12) show the variation of the stress intensity factor K_1 across the thickness of the specimen for two different geometries. They are compared with the results obtained by Yamamoto and Sumi [6] who applied a method involving the superposition of analytical and finite element solutions. The problem of compact tension specimen has also been analyzed by Barsoum [7] and Tracey [8] using isoparametric block elements and wedge shape elements respectively. Both elements are derived by assumed displacements which contain the singular terms for K_1 and K_2 but not for K_3. As indicated in Table 2, by using a mesh pattern with fewer degrees-of-freedom than that by Barsoum and Tracey, the present hybrid stress model provides results which are closer to

the solutions by Yamamoto and Sumi. The latter are used here as reference because they were obtained by finer meshes and also contain a complete treatment of the singular behavior at the crack front.

3. Conclusions

Preliminary results obtained by the presently developed special 3-D crack elements are satisfactory. Further studies are needed to evaluate the merits of the various types of elements and to determine the applicable ranges for the geometry of elements. Finite element solutions should be obtained for problems which have exact solutions to be used as reference.

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TABLE 1

Evaluation of Different Types of Element and Superelement

Type A

No. of Nodes	8	8	12	16
No. of β's	24	24	30	45
K_1	.94250	.94241	.98944	1.0031
K_2	1.0325	1.0287	1.0015	.97760
K_3	1.0574	1.0754	.99874	.99988

Type B

No. of Nodes	8	12	12	16
No. of β's	24	45	51	51
K_1	1.0126	.99841	1.0040	1.0040
K_2	1.0279	.99461	1.0012	1.0012
K_3	1.0019	.99896	1.0029	1.0029

Half Element (Type A + Type B)

Total No. of Nodes	12		20	
No. of Nodes	8(Type 1)	8(Type 2)	12(Type 1)	12(Type 2)
No. of β's	24	24	30	45
K_1	.98245		.99608	
K_2	1.0136		.99647	
K_3	1.0257		.99886	

Table 1 (continued)

Half Element (Type A' + B')

Total No. of Nodes	12	
No. of Nodes	8(Type B')	8(Type A')
No. of β's	24	24
K_1	.96286	
K_2	1.0497	
K_3	1.0257	

Complete Super Element (Type A + B + A' + B')

Total No. of Nodes	20		36	
No. of Nodes	8(Type 1,3)	8(Type 2,4)	12(Type 1,3)	12(Type 2,4)
No. of β's	24	24	30	45
K_1	.98025		.99312	
K_2	1.0290		.99619	
K_3	1.0257		.99886	

TABLE 2 COMPARISON OF 3-D FINITE ELEMENT MODELS FOR THE COMPUTATION OF STRESS INTENSITY FACTOR K_1 OF A COMPACT TENSION SPECIMEN

Author(s) (Year of Publication)	Method of Analysis	Mesh Subdivision	Accuracy of Solution	Remarks
Yamamoto and Sumi (1976)	Superposition of analytical and finite element solution	480 elements 702 nodal pts. 2106 d.o.f.	Calculated K_c* is 6% above K_{2D}**, but with plane strain constraints, it coincides with K_{2D}	To solve on 3-D problem, computation of finite element solution should be performed (2m+1) times, where m is the number of finite element layers which subdivide the plate thickness. Application is limited to simple geometry.
Barsoum (1976)	Displacement method by isoparametric element with four mid-side nodes at the quarter points	150 elements 851 nodal pts. 2553 d.o.f.	K_c is within 2% of K_{2D}, which means up to 10% derivation from Yamamoto's solution	Stress intensity factor is calculated from the displacement solution for nodes closest to the crack tip. The element lacks the mode 3 singularity. The isoparametric element is a part of the element library of most general purpose programs, so its use is tractable.
Tracey (1974)	Displacement model by wedge shape element with square root r displacement variation	522 elements 660 nodal pts. 1980 d.o.f.	K_c is 1% below K_{2D}, which means 9% deviation from Yamamoto's sol. With plane strain constraint, K_c is 6% below K_{2D}	The element lacks the constant strain and the rigid body motion modes and also lacks the mode 3 singularity. Stress intensity factor is inferred from the opening displacement of the crack face.
Present hybrid stress model including asymptotically exact stresses and boundary displacements assumption		85 elements 168 nodal pts. 504 d.o.f.	K_c is within 3% below of Yamamoto's solution	Stress intensity factors are directly computed. The element contains all singular modes. The accuracy is very high even when the very coarse mesh subdivision is used.

*K_c: stress intensity factor K_1 at the center of the specimen

**K_{2D}: 2-dimensional plane strain stress intensity factor solution

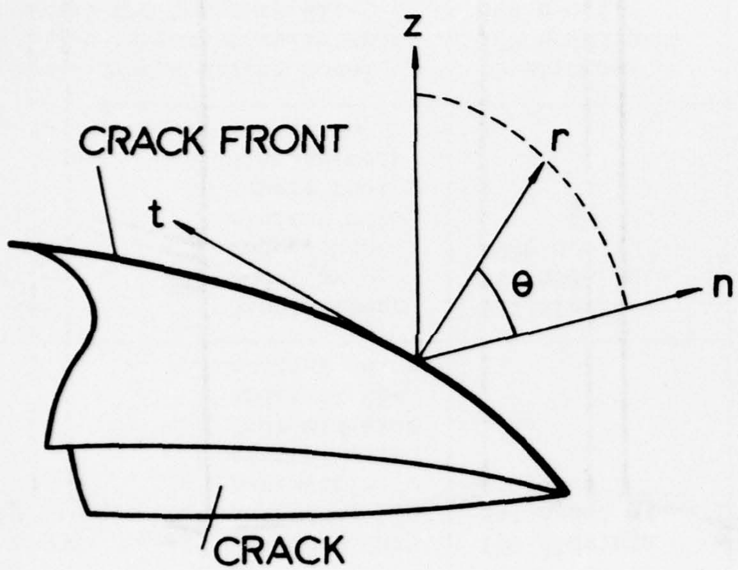
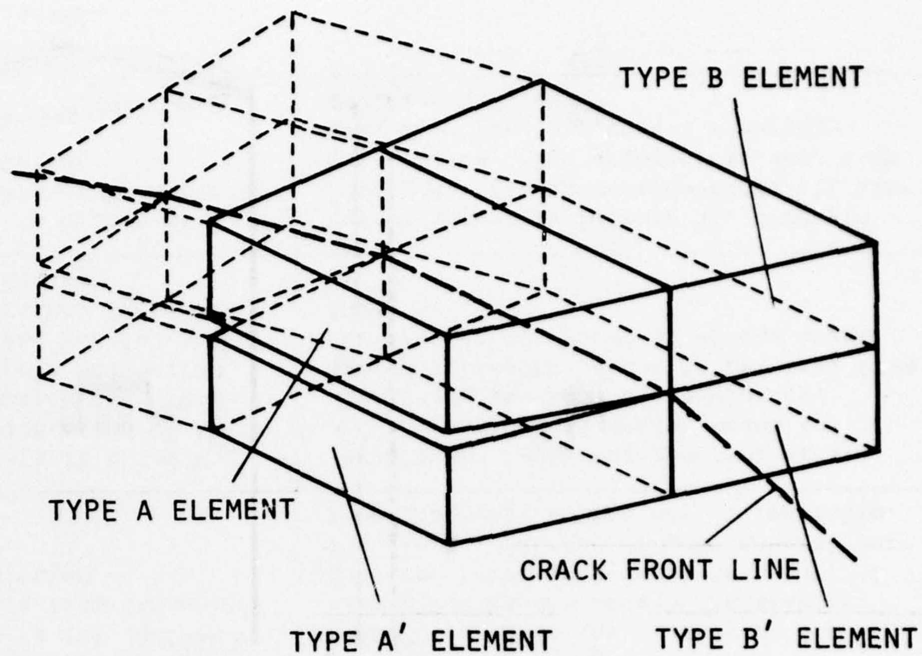


FIG. 1 LOCAL COORDINATES SYSTEM



TYPE A ELEMENT : SINGULAR HYBRID STRESS ELEMENT WITH STRESS FREE CONDITION OVER CRACK SURFACE

TYPE B ELEMENT : SINGULAR HYBRID STRESS ELEMENT

FIG. 2 FINITE ELEMENT IDEALIZATION AROUND CRACK FRONT

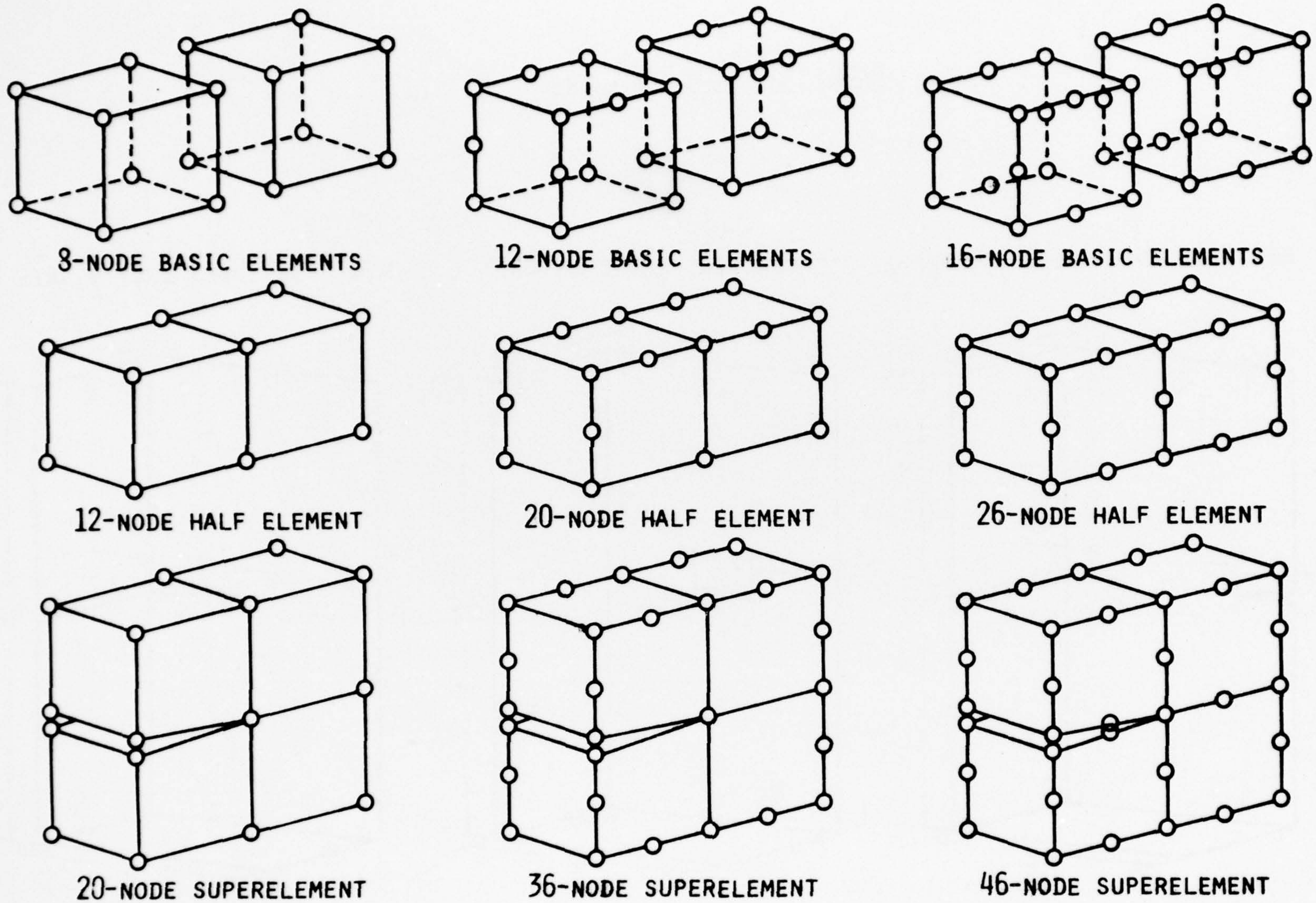


FIG. 3 FAMILY OF THREE DIMENSIONAL HYBRID STRESS CRACK ELEMENTS

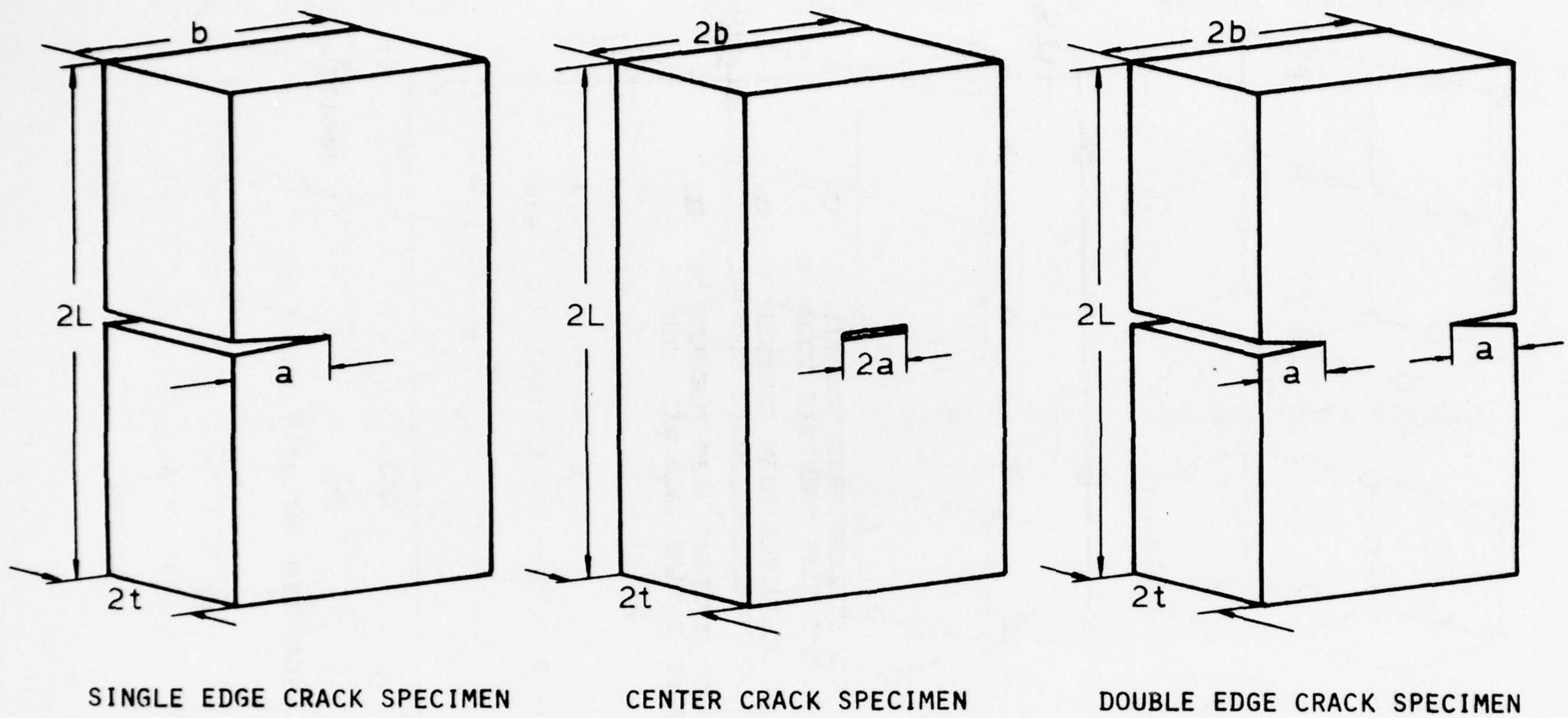
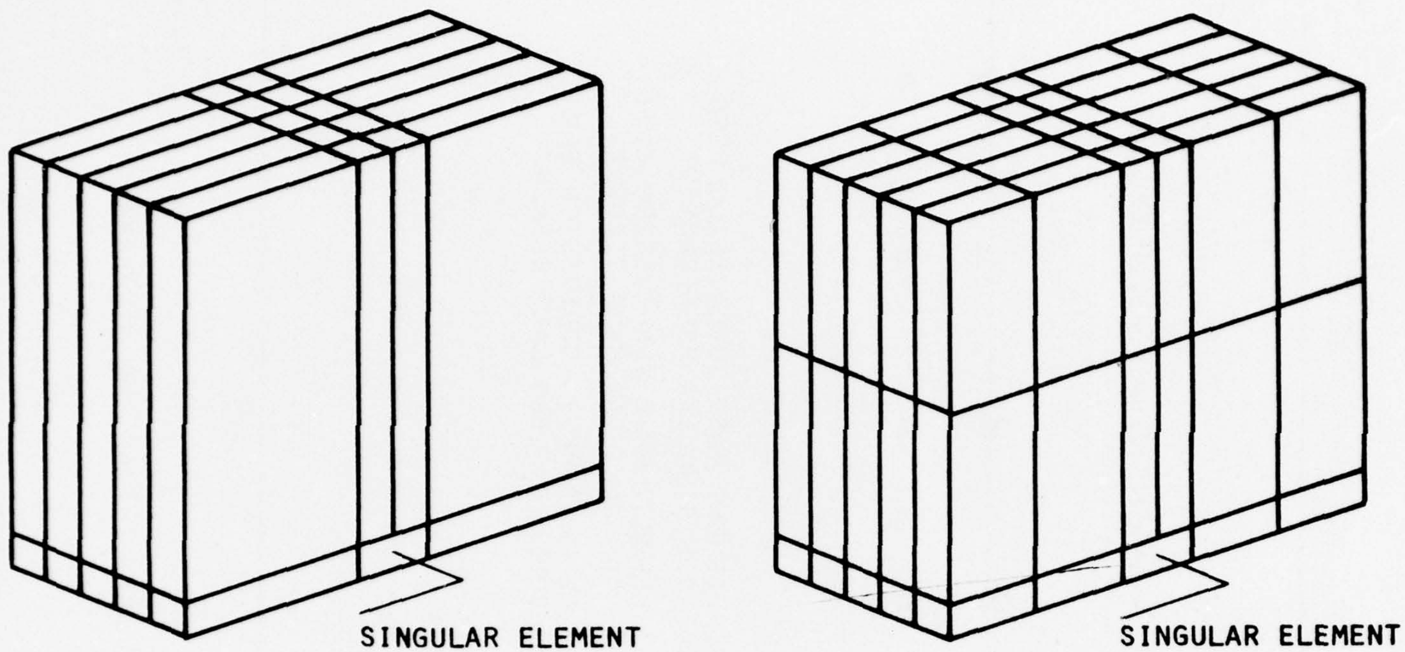


FIG. 4 FRACTURE TEST SPECIMENS



(1) 35 ELEMENTS (2) 85 ELEMENTS

FIG. 5 FINITE ELEMENT BREAKDOWN FOR THE ANALYSIS OF FRACTURE TEST SPECIMENS

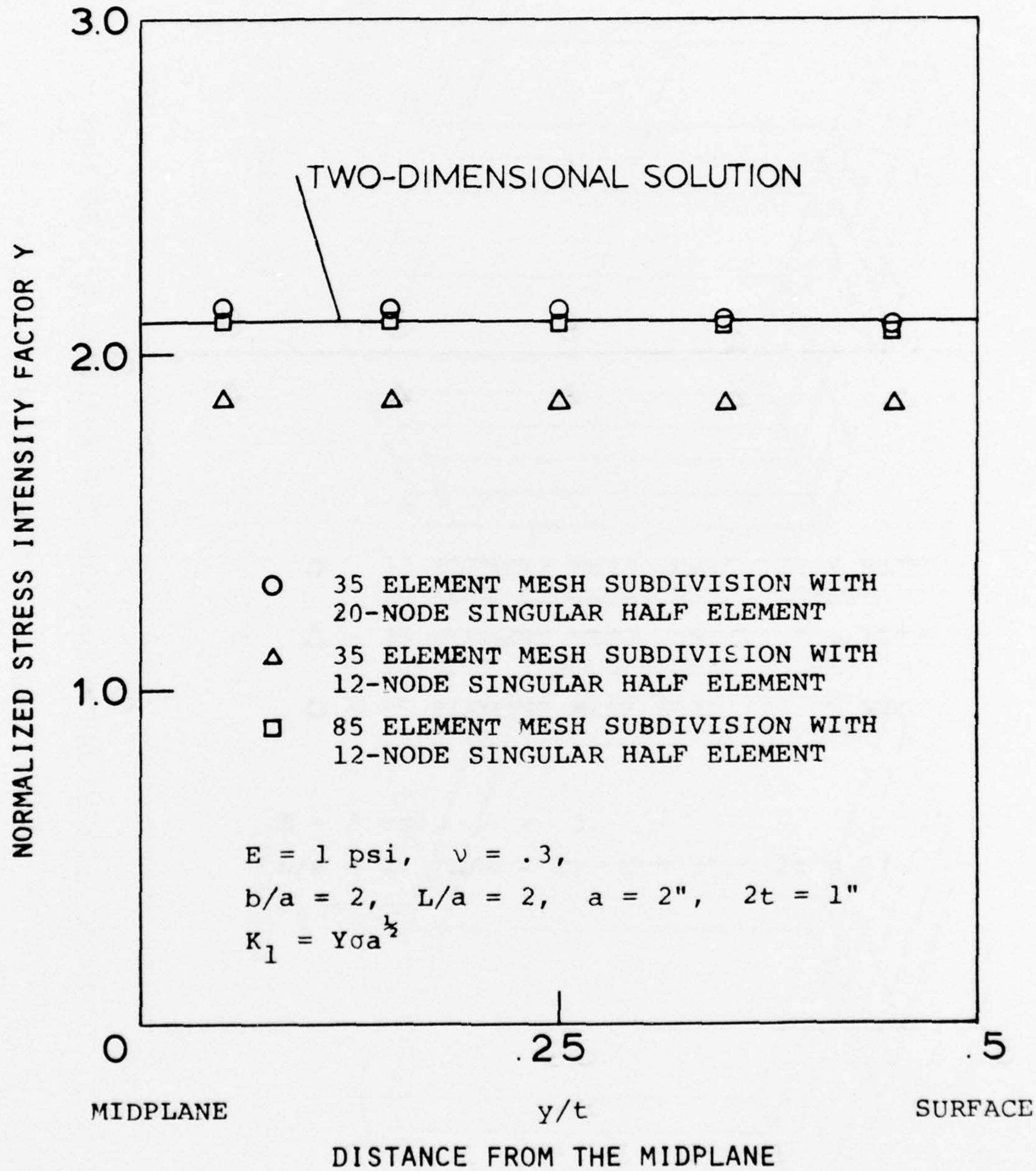


FIG. 6 CENTER CRACK SPECIMEN

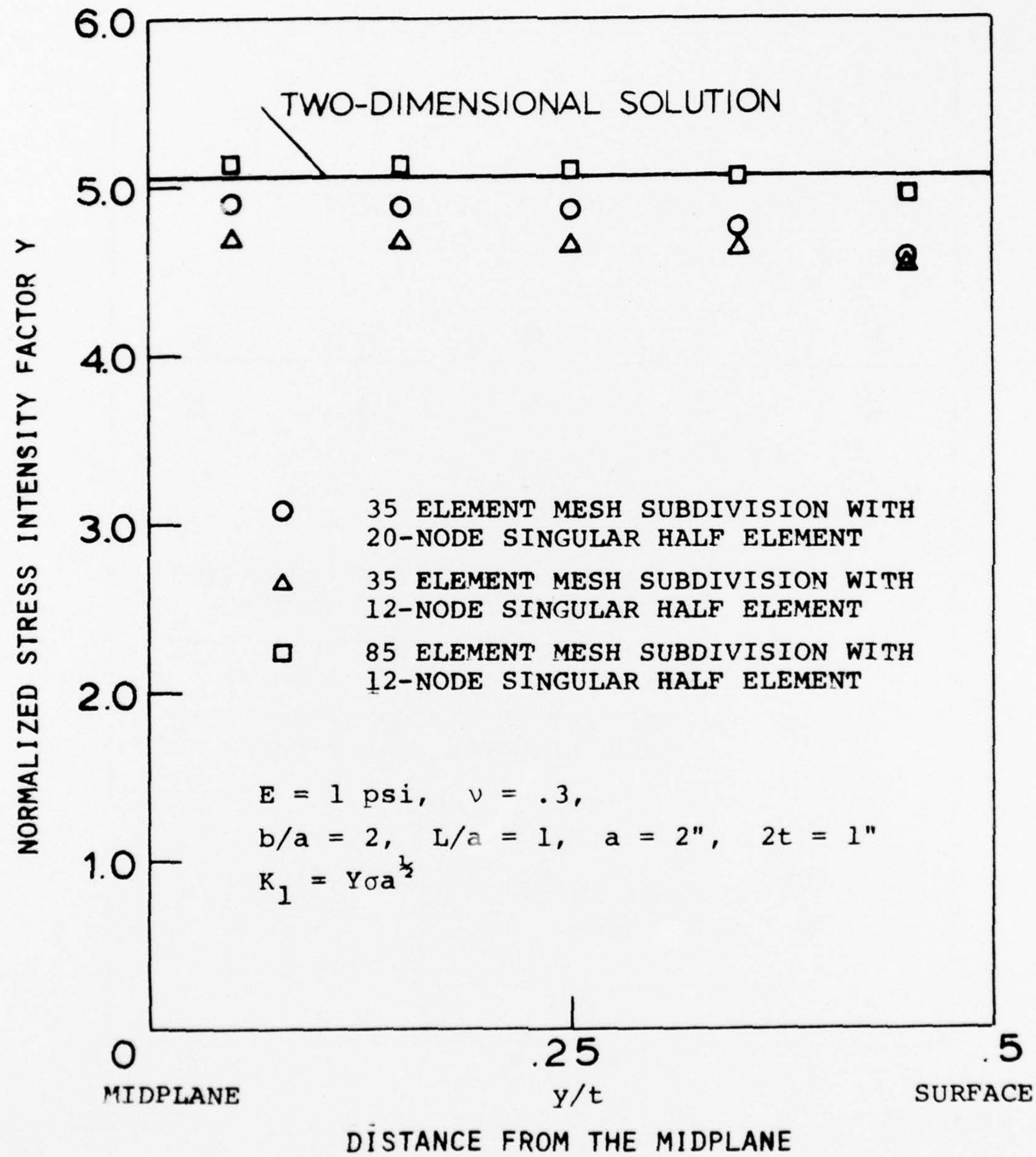


FIG. 7 SINGLE EDGE CRACK SPECIMEN

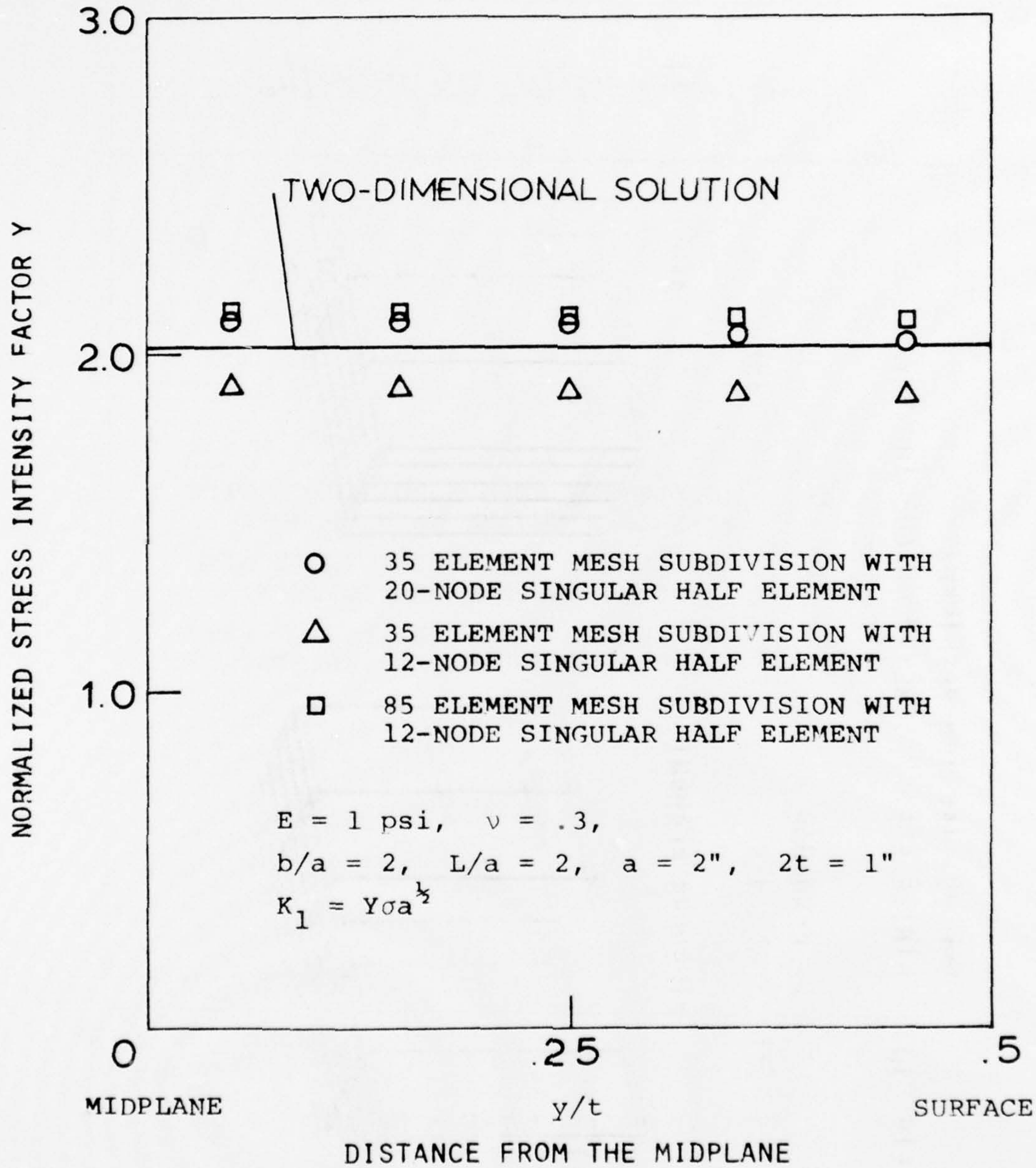


FIG. 8 DOUBLE EDGE CRACK SPECIMEN

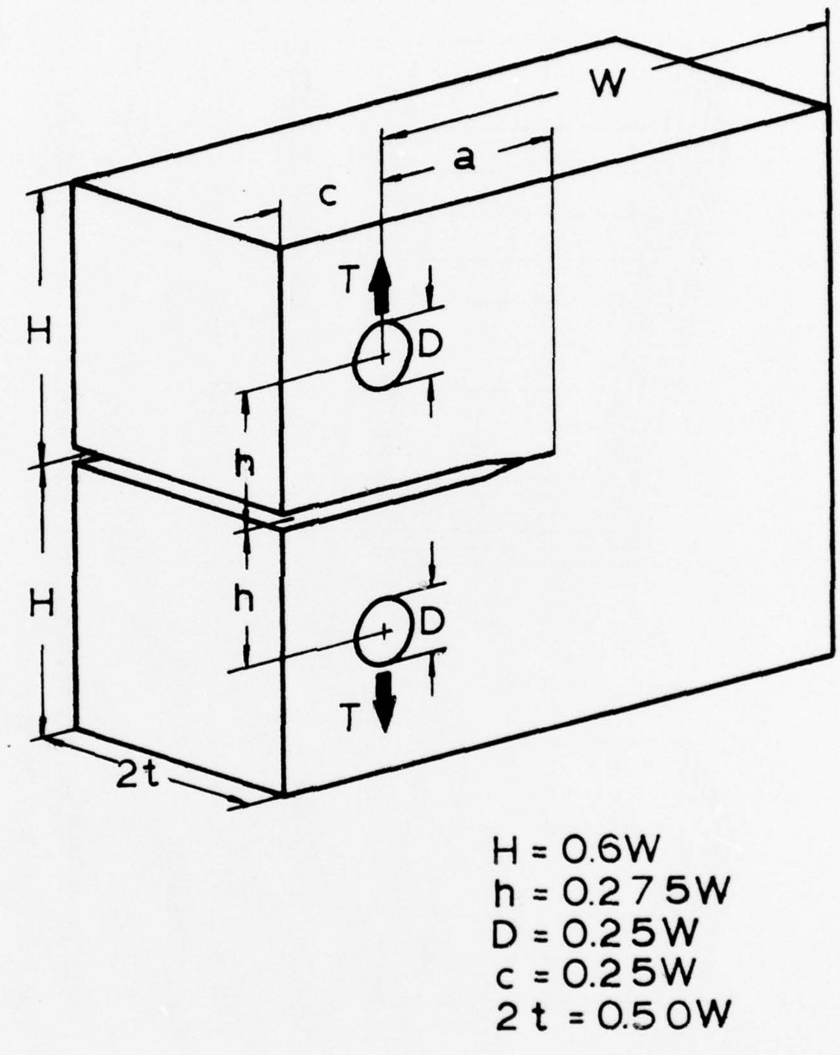
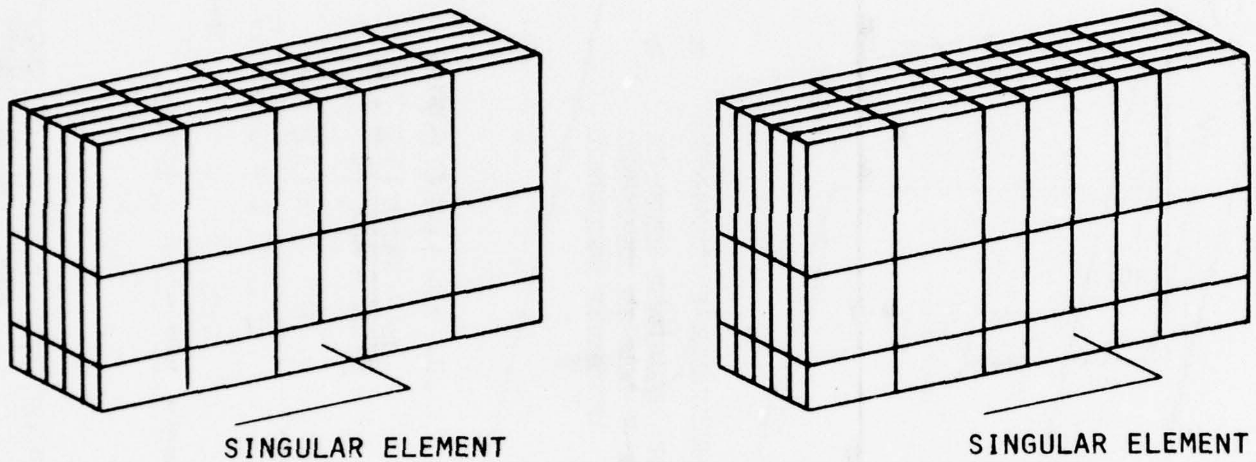


FIG. 9 STANDARD COMPACT TENSION SPECIMEN (ASTM STANDARD E-399-72)



(1) a/W = 3/8
85 ELEMENTS

(2) a/W = 5/8
105 ELEMENTS

FIG. 10 FINITE ELEMENT BREAKDOWN FOR THE ANALYSIS OF STANDARD COMPACT TENSION SPECIMENS

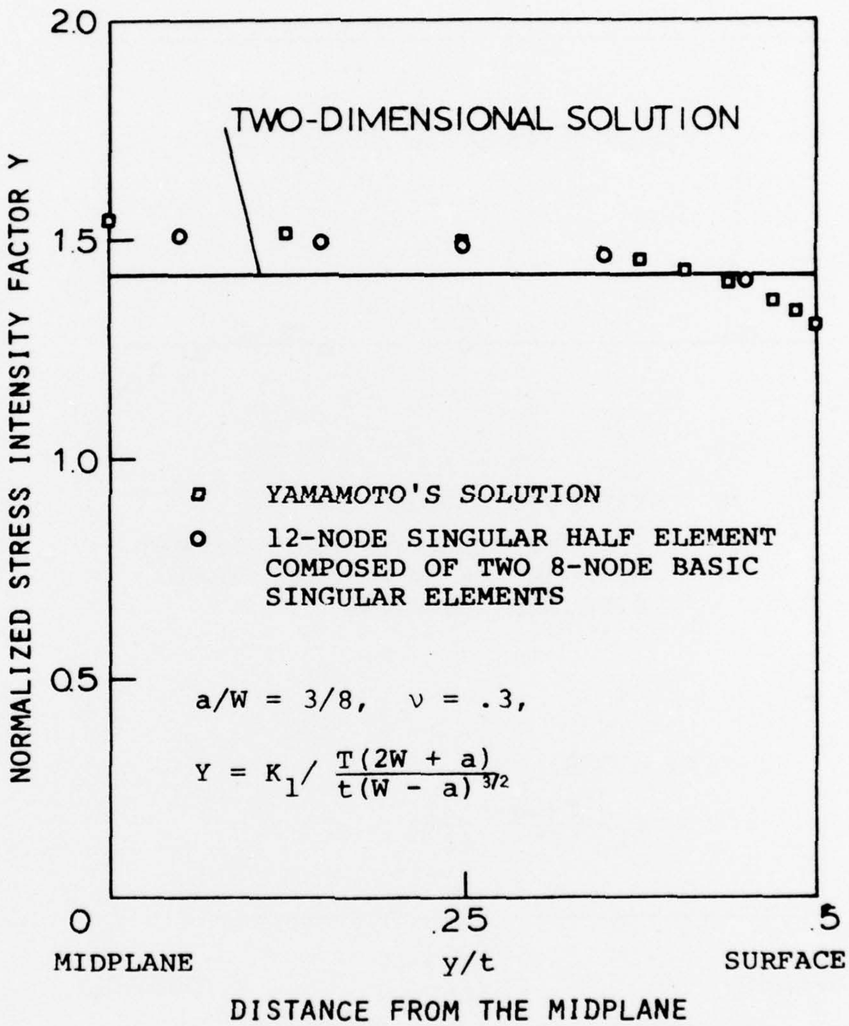


FIG. 11 STRESS INTENSITY FACTOR FOR COMPACT TENSION SPECIMEN

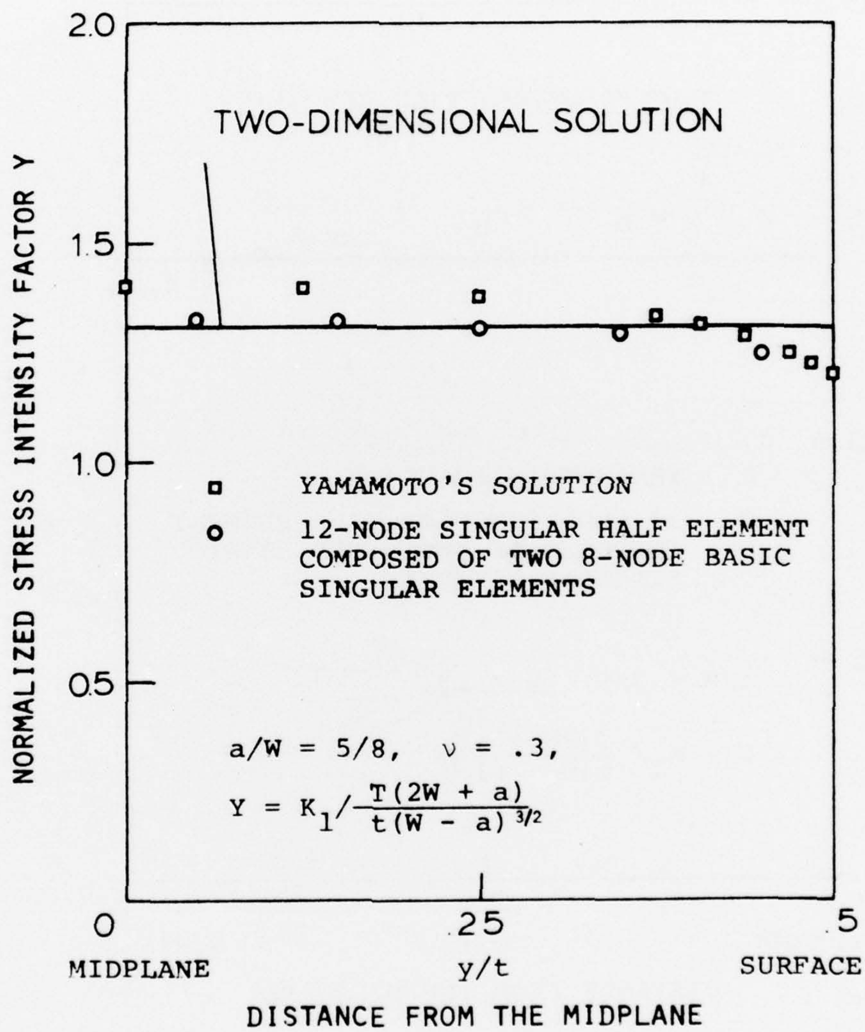


FIG. 12 STRESS INTENSITY FACTOR FOR COMPACT TENSION SPECIMEN